THE MULTIPLICITY OF EIGENVALUES
IN THE ADJACENCY MATRIX OF A TREE

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Overview

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Previous Results on the Multiplicity of Eigenvalues of a Matrix Whose Graph Is a Tree

**Johnson and Duarte** – The maximum possible multiplicity of any eigenvalue is bounded by the minimum number of vertex disjoint paths occurring as induced subgraphs of the tree that cover all of the vertices of the tree. (Result for a general class of real \( n \times n \) symmetric matrices.)

**Cvetkovic and Gutman** – The exact multiplicity of the eigenvalue zero in the adjacency matrix of a tree is equal to the difference of the number of vertices in the tree and twice the cardinality of the maximum matching. This is equivalent to the number of unmatched vertices.
Our Main Result

Gardner and Leganza – The exact multiplicity of any eigenvalue $\lambda$ can be specified in terms of matchings of special subgraphs of the tree, called minimal graphs, whose adjacency matrix has $\lambda$ as an eigenvalue. Specifically, the multiplicity is the number of unmatched minimal graphs.
A matching $M \subseteq E$ is a subset of edges with the property that each vertex in the subgraph of $G$ induced by $M$ has no more than one incident edge. [Nemhauser and Wolsey, pp.609-613]

A maximum cardinality matching.

\[ N(0) = |V| - 2|M_{max}| = 7 - 2(2) = 3. \]
Applying the Matching Concept
To Find the Multiplicity of Any Eigenvalue

1. Find the smallest tree having the eigenvalue $\lambda$

2. Find subgraphs isomorphic to this smallest tree in a larger tree, and

3. Match the subgraphs to vertices and count how many left over.
The smallest tree having 1 and –1 as eigenvalues.

Tree with \( N(1)=2 \)

Second tree with \( N(1)=2 \)
Tree with $N(0)=2$ from which the graphs are constructed.
Minimal Graphs

The smallest tree having 1 and \(-1\) as eigenvalues.

Another tree 1 and \(-1\) as eigenvalues.

Matchings of the first tree in the second indicate that the multiplicity of 1 and \(-1\) should be zero for the second.
We can construct trees with both of these graphs having the right multiplicity of the eigenvalues, $N(1)=N(-1)=1$.

Second configuration.
Definition of Minimal Graph

Definition. A tree, $G$, is minimal with respect to the eigenvalue $\lambda$ if one of the following conditions holds:

1. $G$ has the eigenvalue $\lambda$ but has no proper subgraph having the eigenvalue $\lambda$, or

2. $G$ has the eigenvalue $\lambda$ and $N(\lambda) > \max[p-q]$ over all nonempty deleted subsets $V'$ of the vertex set of $G$ where $q = |V'|$, and where $p$ is the sum of the multiplicities of the eigenvalue $\lambda$ over the components left after deletion of $V'$ that have the eigenvalue $\lambda$ and that have fewer vertices than $G$. 
Minimal Graphs - Examples

Smallest tree having the eigenvalues $\pm \sqrt{2}$.

Tree minimal with respect to the eigenvalues $\pm \sqrt{2}$. 
Putting Minimal Graphs Together

Example of tree with $N(\pm \sqrt{2})=1$.

Example of tree with $N(\pm \sqrt{2})=2$. 
Minimal Graphs - Examples

Smallest tree having the eigenvalues $\pm \sqrt{3}$.

Tree minimal with respect to the eigenvalues $\pm \sqrt{3}$. 
Putting Minimal Graphs Together

Example of tree with $N(\pm \sqrt{3})=1$.

Second example of tree with $N(\pm \sqrt{3})=1$. 
Open Questions

If a tree $G$ contains a minimal graph with respect to $\lambda$ as a subgraph that is adjacent to a vertex whose deletion leaves no other subgraphs minimal with respect to $\lambda$, why in some cases does $G$ have $\lambda$ as an eigenvalue and other cases not?

(We know from the proof of the main theorem that a graph minimal with respect to the eigenvalue $\lambda$ cannot have two or more subgraphs that are also minimal with respect to $\lambda$ adjacent to the same vertex.)

Can a graph that is minimal with respect to $\lambda$ have $N(\lambda)>1$?
The Main Theorem

**THEOREM.** Let $G$ be a tree that is not minimal with respect to the eigenvalue $\lambda$. Then $N(\lambda)=\max[p-q]$ over all subsets $V'$ of the vertices of $G$ where $q=|V'|$, whose deletion leaves $n$ components that are minimal with respect to the eigenvalue $\lambda$ having fewer vertices than $G$ such that the sum of the multiplicity of the eigenvalue $\lambda$ in those $n$ graphs is $p$. 
Highlights of Our Proof

$G$ not minimal with respect to $\lambda$
subgraphs $X$ and $Y$ have $\lambda$ as an eigenvalue

\[ C = \begin{bmatrix} C_X \\ 0 \\ kC_Y \end{bmatrix} \]
is an eigenvector of $G$ corresponding to $\lambda$
where $C_X$ is an eigenvector of $X$ corresponding to $\lambda$,
$C_Y$ is an eigenvector of $Y$ corresponding to $\lambda$ and
$k = -c_{xi}/c_{yj}$ where $i$ is the vertex adjacent to $v$ in $X$ and
$j$ is the vertex adjacent to $v$ in $Y$. 
The Main Idea of the Proof

• For a symmetric graph, the number of linearly independent eigenvectors corresponding to an eigenvalue is the multiplicity of the eigenvalue.

• We want to show that if a set of subgraphs each having \( \lambda \) as an eigenvalue are adjacent to a single vertex, then multiplicity of \( \lambda \) in the graph is one less than the sum of the multiplicities of the eigenvalues in the subgraphs.

• This allows us to use matching to find the multiplicity of the eigenvalue.
Subgraphs with Multiplicity of \( \lambda \) Greater than One

Let \( C_X^1 \) and \( C_X^2 \) be linearly independent eigenvectors of \( X \)

\( C_Y^1 \) and \( C_Y^2 \) be linearly independent eigenvectors of \( Y \)

\[
\begin{bmatrix}
C_X^1 \\
0 \\
k_{11} C_Y^1 \\
\end{bmatrix} + a_2
\begin{bmatrix}
C_X^1 \\
0 \\
k_{12} C_Y^2 \\
\end{bmatrix} + a_3
\begin{bmatrix}
C_X^2 \\
0 \\
k_{21} C_Y^1 \\
\end{bmatrix} =
\begin{bmatrix}
C_X^2 \\
0 \\
k_{22} C_Y^2 \\
\end{bmatrix}
\]

if \( a_1 = -k_{21}/k_{11}, \) \( a_2 = k_{22}/k_{12}, \) and \( a_3 = 1. \)

Note that \( k_{ij} = -c_x^i/c_y^j \) for each \( i, j \in \{1,2\} \) and

\( k_{21}/k_{11} = k_{22}/k_{12} = c_x^2/c_x^1. \)
Multiple Subgraphs with $\lambda$ as an Eigenvalue

\[
\begin{align*}
X_1 & \xrightarrow{v} X_2 \\
X_2 & \xrightarrow{v} X_3 \\
& \cdots \\
X_n & \xrightarrow{v} X_1
\end{align*}
\]

\[
a_1 \begin{bmatrix}
0 \\
C_1^1 \\
k_{12}C_2^1 \\
0
\end{bmatrix}
+ a_2 \begin{bmatrix}
0 \\
0 \\
C_2^1 \\
k_{23}C_3^1 \\
0
\end{bmatrix}
+ \ldots + a_{n-1} \begin{bmatrix}
0 \\
0 \\
C_{n-1}^1 \\
k_{n-1,n}C_n^1 \\
0
\end{bmatrix}
+ a_n \begin{bmatrix}
0 \\
0 \\
k_{n1}C_1^1 \\
0 \\
C_n^1
\end{bmatrix}
= 0.
\]

if we let $a_i=1/c_i^1$. 
Conclusions and Future Directions

The matching concept allows us
  • to find the multiplicity of an eigenvalue in terms of subgraphs of a tree, and
  • to construct trees with a given multiplicity of an eigenvalue.

Future research
  • Open questions on the structure and multiplicities of eigenvalues in minimal graphs
  • Expand to other types of matrices that represent trees
  • Expand concept to matroids and help us identify the number of triads, triangles and other structures in matroids.