What follows is my paper "Geometry and proof in Abū Kāmil's algebra". Pp. 234-256 in: *Actes du 10$^{ème}$ Colloque Maghrébin sur l'Histoire des Mathématiques Arabes (Tunis, 29-30-31 mai 2010).* Tunis: L'Association Tunisienne des Sciences Mathématiques, 2011.

The published version was adjusted to fit the book, so one cannot simply add 233 to the page numbers of this version to get the published version.

If you have any questions or comments, send them to me: oaks@uindy.edu .

—Jeff Oaks

Geometry and proof in Abū Kāmil's algebra

Jeffrey A. Oaks[1]

## 1. Introduction

Scholars in medieval Islam were fortunate in having at their disposal sophisticated scientific knowledge from a number of earlier cultures, most notably Greece, Persia and India. One way that mathematicians and others took advantage of this diverse pool was to combine elements from different traditions. Abū Kāmil's late ninth century *Book of Algebra* is a good example. His application of Greek-style proof to practical Arabic arithmetic and algebra is particularly interesting because it provides testimony to both the success and the difficulties of the merging of ideas.

Among Arabic algebra books, Abū Kāmil's stands out for the large number of proofs—fifty in all—found throughout the book. He builds on al-Khwārizmī's treatment by giving fifteen proofs for different rules to solve simplified equations. The remaining proofs are distributed across various settings: for multiplying monomials and binomials, for operating with square roots, for the solutions to entire problems, for the rule for setting up a polynomial equation in some problems, and for propositions in arithmetic.

All but one of Abū Kāmil's proofs are accompanied by some kind of diagram. These proofs come in two kinds, which I call geometrical and arithmetical. In a geometrical proof the operations of arithmetic are reinterpreted as operations in geometry. In particular, the multiplication of two quantities is represented in the proof by the formation of a rectangle. In an arithmetical proof no real construction takes place. The

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[1] University of Indianapolis, oaks@uindy.edu. Note: The Diagrams were generated using DRaFT (http://www.hs.osakafu-u.ac.jp/~ken.saito) from the reproduction of the Istanbul MS in [Abū Kāmil 1986]. I thank Marco Panza initiating my interest in algebra proofs.

operations remain in the realm of arithmetic, and the disjoint lines comprising the diagram serve only to represent and label arbitrary numbers. The result of multiplying the number represented by line *A* by the number represented by the line *B*, for instance, is the number represented by the new line *G*.

The choice of which kind of proof to present is not arbitrary. Abū Kāmil gives geometrical proofs for rules stated in terms of specific examples, such as the solution to $x^2 + 10x = 39$ and for calculations like $(10 - x) \cdot (10 - x) = 100 + x^2 - 20x$ and $\sqrt{4} + \sqrt{9} = 5$. With only one exception he gives arithmetical proofs to propositions about arithmetic stated in general terms, like the rule $\frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab}$. A likely reason for Abū Kāmil's dual approach lies with the practices of his two main influences, al-Khwārizmī and Euclid. Al-Khwārizmī gave geometrical proofs for the rules to solve specific equations, while Euclid gave arithmetical proofs to general propositions in his books on number theory.

There is a disparity between Arabic arithmetic and both Greek geometry and Greek arithmetic which causes Abū Kāmil to compromise with both kinds of proof. Numbers, whether Arabic or Greek, are closed under multiplication and division, while magnitudes change dimension under these operations. So in his geometrical proofs Abū Kāmil relaxes his standard of rigor by identifying arithmetical and geometrical operations. But he also benefits from this, because a geometrical interpretation allows him to rest his arguments on propositions from the geometrical books of the *Elements*.

Euclid's numbers consist only of positive integers, so his propositions deal with elementary number theory. Arabic numbers include fractions and irrational roots, and Abū Kāmil's propositions are rules of calculation. So although Abū Kāmil gains some rigor when he opts for an arithmetical proof, he also loses the possibility grounding his proofs in propositions from the *Elements*.

Abū Kāmil violates his dual approach for one arithmetic result stated in general terms. In his proof to the general rule $a - n\sqrt{a} = b + n\sqrt{b}$ implies $\sqrt{a} = \sqrt{b} + n$ ($n = 1$, 2, etc.) he produces a real geometric construction with rectangles and squares representing the results of multiplications. The proof is rather complex, which may be part of the reason he did not give an arithmetical proof. He then gives a second proof, by algebra (and thus without a diagram), perhaps to make up for this failure. But this is also the only proposition proven by Abū Kāmil which is easily handled by algebra. I have not seen any other proof by algebra in all of medieval mathematics.

## 2. Overview of Abū Kāmil's *Kitāb al-jabr wa'l-muqābala*

Abū Kāmil Shujā' ibn Aslam ibn Muḥammad ibn Shujā' (Abū Kāmil) wrote his *Kitāb fī al-jabr wa'l-muqābala* (*Book on Algebra*, henceforth *Algebra*) around 890 CE. He modelled his book on al-Khwārizmī's famous treatise *Kitāb al-jabr wa'l-muqābala* (*Book of Algebra*), written earlier in the century. Like his predecessor, Abū Kāmil first presents the rules of algebra, and then gives a collection of worked-out problems. Both books are entirely rhetorical. No symbols are used at all, and even the numbers are written out in words.

Abū Kāmil's book begins with a brief explanation of the standard names for the powers of the unknown. Numbers (constants) are usually counted in dirhams (a silver coin). The first power is named *jidhr* ("root") but it is more often called *shay'* ("thing") in the worked-out problems. The second power is called *māl* ("sum of money", "treasure"). Higher powers are also used, but they are not connected with any proofs.

Following al-Khwārizmī, our algebraist then classifies and solves the six types of first- and second-degree equation, giving proofs for types 1, 4, 5, and 6:[2]

[2] There are six types instead of our one $ax^2 + bx + c = 0$ because the rules for solving the equations take the number (coefficients) of the terms as parameters, and medieval mathematicians recognized only positive numbers.

Simple equations

(1) *māl*s equal roots ( $ax^2 = bx$ )

(2) *māl*s equal number ( $ax^2 = c$ )

(3) roots equal number ( $bx = c$ )

Composite equations

(4) *māl*s and roots equal number ( $ax^2 + bx = c$ )

(5) *māl*s and number equal roots ( $ax^2 + c = bx$ )

(6) roots and number equal *māl*s ( $bx + c = ax^2$ )[3]

He then gives rules and proofs for multiplying monomials and binomials, which is followed by instructions, again with proofs, on how to operate with square roots.

After presenting these rules Abū Kāmil states and solves 74 problems. The first six problems are designed to illustrate the solutions to the six types of equation. I number these (T1) to (T6). After the algebraic solution to each problem he gives a proof by geometry that the solution is valid. These problems are followed by sixty-eight more problems, which I number (1) to (68).[4] Abū Kāmil proves different kinds of propositions even within the solutions to these problems.

## 3. Abū Kāmil's proofs

The proofs in the *Kitāb fī al-jabr wa'l-muqābala* fall naturally into six categories:

Proofs in the first part, on the rules of algebra

(1) The rules for solving simplified equations (15 proofs to 10 rules)

(2) Products of monomials and binomials (8 proofs)

(3) Calculations with roots of numbers (8 proofs)

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[3] [Abū Kāmil 1986, 5;3, 7;2; 2004, 18;20, 20;14].

[4] References to the problems are listed in Appendix A in [Oaks & Alkhateeb 2005].

Proofs in the second part, the worked-out problems

(4) The algebraic solutions to problems (T1) – (T6) (6 proofs)

(5) The rules to set up polynomial equations in problems (4) – (7) (4 proofs)

(6) Arithmetic theorems used in problems (2), (7), (61), (62), and (63) (9 proofs to 8 rules)

A list of the propositions in each category is given in Appendix A.

The propositions—mainly rules and calculations—are either stated generally, or are expressed in terms of a specific example. Three propositions in category (3) and all the propositions in category (6) are stated in general terms. For instance, one theorem in arithmetic proven in problem (2) is expressed as: "For any two numbers, you divide one of them by the other. So the result of the division is equal to the result of dividing the product of the divisor by itself, by the product of the divisor by the dividend"[5] $(\frac{a}{b}=\frac{a^2}{ab})$. The proof to this proposition is arithmetical. The two numbers are represented by lines labeled *A* and *B*, and among other calculations, the product of *A* by *B* is a new line *D*.

The other propositions are framed in terms of specific examples, but the rules are stated so that the general rule is evident. For example, the rule for finding the "thing" ($x$) in the equation "a *māl* and ten roots equal thirty-nine dirhams" $(x^2+10x=39)$ can be followed for any type 4 equation. The multiplication of "ten and a thing" by "ten less a thing" is explained in such a way that one can see how to multiply any similar binomials, and the sum of $\sqrt{9}$ and $\sqrt{4}$ is worked out according to the rule $\sqrt{a}+\sqrt{b}=\sqrt{(a+b)+2\sqrt{ab}}$. The proofs to these rules are all geometrical. In the last mentioned example the product of $\sqrt{4}$ by $\sqrt{9}$ is represented in the diagram by a rectangle whose sides have lengths $\sqrt{4}$ and $\sqrt{9}$.

[5] [Abū Kāmil 1986, 51;20; 2004, 83;1].

With one exception from category (6), all proofs to general propositions are arithmetical, and all proofs to specific calculations are geometrical.

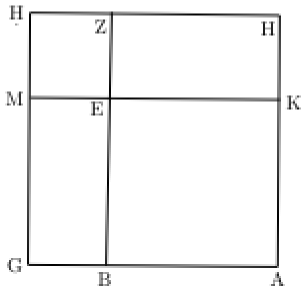
## 4. Geometrical and arithmetical proofs

The difference between geometrical proofs of specific calculations and arithmetical proofs of general rules is clearest in the following two proofs of the same rule, first as $\sqrt{9}\cdot\sqrt{4}=\sqrt{9\cdot 4}=6$ and then as $\sqrt{a\cdot b}=\sqrt{a}\cdot\sqrt{b}$:[6]

> So if you wanted to multiply the root of nine by the root of four, so multiply nine by four, so it yields thirty-six. So you take its root, which is six, which is the root of nine by the root of four.
>
> And I show this by this figure, which is that we make line *AB* the root of nine, and line *BG* the root of four. And if we wanted to multiply line *AB* by line *BG*, we construct on line *AG* the square surface *AḤḤ*. And we draw from point *B* a line parallel to lines *AH* and *GḤ*, which is line *BZ*. And each of the lines *AH*, *GḤ* is the root of nine and the root of four. So we make line *MḤ* the root of four and we draw from point *M* a line parallel to lines *AG*, *ḤH*, which is line *MK*. So surface *AE* is nine and line *KE* is the root of nine, and surface *EḤ* is four, and line *ZE* is the root of four, since line *ZE* is equal to line *EM*, and line *EM* is equal to line *BG*, and *EB* is equal to *EK*.
>
> And the ratio of *ME* to *EK* is as the ratio of *ZE* to *EB*. And the ratio of *ME* to *EK* is as the ratio of surface *EḤ* to surface *ZK*. And the ratio of *ZE* to *EB* is as the ratio of surface *ZK* to surface *EA*. So the ratio of surface *ZM* to surface *ZK* is as the ratio of surface *ZK* to surface *EA*.
>
> So the product of the number that corresponds to surface *ZM* by the number that corresponds to surface *EA* is equal to the number

[6] [Abū Kāmil 1986, 35;16; 2004, 59;8].

that corresponds to surface *ZK* by itself.[7] And Euclid showed this in Book VI of his work.[8] He said that for three proportional numbers the product of the first number by the third number is equal to the product of the second number by itself. So we multiply what is in surface *ḤE* in units, which is four, by what is in surface *AE* in units, which is nine, so it yields thirty-six. So the product of what is in surface *ZK* in units by itself is thirty-six. So surface *ZK* is the root of thirty-six, which is six, which comes from the multiplication of the root of nine by the root of four, since *KE* is the root of nine and *EM* is the root of four. And that is what we wanted to show.




*Abū Kāmil then calculates a couple examples:* $\sqrt{10}\cdot\sqrt{3}=\sqrt{30}$ and $2\sqrt{10}\cdot\frac{1}{2}\sqrt{5}=\sqrt{50}$. *The format of this next proof follows that of Euclid, so I name the parts.*

And I write this rule in a general form. And there is no power except by God, the exalted and great.

[*protasis* (enunciation)]

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[7] As geometric magnitudes these products would be four dimensional, so Abū Kāmil shifts the calculation back to the numbers they represent.

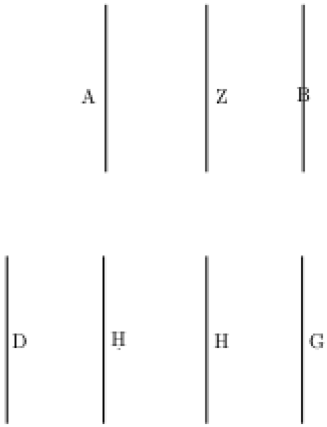
[8] *Elements* VI.17. This proposition is about geometric magnitudes. Note that he did not to appeal to Proposition VII.19, which states for numbers that $A:B=C:D$ if and only if $AD=BC$.

One multiplies a number by a number, then one takes the root of what results. So it yields the same as the product of the root of one of the numbers by the root of the other number.

[*ekthesis* (setting out)]
An example of this is that *B* is a number and *A* is a number. And the root of *B* is *G*, and the root of *A* is *D*. And the product of *B* by *A* gives *Z*, and the root of *Z* is *H*. And the product of *G* by *D* is *Ḥ*.

[*diorismos* (definition of goal)]
So I say that *Ḥ* is equal to *H*.

[*apodeixis* (proof)]
And the proof of this is that *G* multiplied by itself gives *B*, and multiplied by *D* gives *Ḥ*. So the quantity *G* is to *D* as the quantity *B* is to *Ḥ*. And also *D* multiplied by itself gives *A*, and multiplied by *G* gives *Ḥ*. So the quantity *G* is to *D* as the quantity *Ḥ* is to *A*. And already the quantity *jīm* [*G*] is to the *dāl* [*D*] as the quantity *B* is to *Ḥ*. So the product of *B* by *A* is equal to the product of *Ḥ* by itself. And the product of *B* by *A* is *Z*. So the product of *Ḥ* by itself is *Z*. And the product of *H* by itself is *Z*. So the product of *Ḥ* by itself is equal to the product of *H* by itself. So *Ḥ* is equal to *H*, and that is what we wanted to show.




The two proofs are logically equivalent, though the geometrical proof is easier to follow than the arithmetical proof.

Keep in mind that although most of Abū Kāmil's rules are stated in terms of numbers, they are applied to algebraic quantities in the worked-out problems. This particular proposition is used in problems (33), (34), (54), and (55). One example from problem (55) is "So multiply the root of half a thing by the root of a third of a thing, so it yields the root of a sixth of a *māl*"[9] ($\sqrt{\frac{1}{2}x} \cdot \sqrt{\frac{1}{3}x} = \sqrt{\frac{1}{6}x^2}$).

## 5. Answering to two masters: al-Khwārizmī and Euclid

The two biggest influences on Abū Kāmil's *Algebra* are al-Khwārizmī's *Algebra* and Euclid's *Elements*. Abū Kāmil wrote his book as a commentary on al-Khwārizmī's, so he follows al-Khwārizmī in overall organization, and he carries over many specific equations and problems from him as well. The *Elements* is the source for the format of many of

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[9] [Abū Kāmil 1986, 104;1; 2004, 167;9].

Abū Kāmil's proofs, and he makes references to several of Euclid's propositions within his geometrical proofs.

The ways al-Khwārizmī and Euclid composed their proofs may explain Abū Kāmil's dual approach. Al-Khwārizmī gave geometrical proofs to the rules for specific equations, while Euclid gave arithmetical proofs to his general propositions on number theory. However, in other aspects the influences of these mathematicians on each other is not so simple. I describe presently these influences in more detail, after which I turn to Abū Kāmil's arithmetical proofs and the incompatibility of Arabic arithmetic with both Greek geometry and Greek arithmetic. I then offer some thoughts about why Abū Kāmil wrote two kinds of proof.

The proofs in Euclid's *Elements* ideally contain six parts: the *protasis* (enunciation), *ekthesis* (setting-out), *diorismos* (definition of goal), *kataskeu\=e* (construction), *apodeixis* (proof), and *superasma* (conclusion).[10] Abū Kāmil adopted the same format for his arithmetical proofs, and also for some of his geometrical proofs.[11] Just as in Euclid, Abū Kāmil's parts are not named explicitly, but are often marked by formulaic phrases such as "An example of this…" (setting-out), "So I say…" (definition of goal), "the proof of this…" (proof), and "we have explained that…" (conclusion). They are also occasionally marked in the manuscript by a punctuation symbol, which resembles a "@", but with a dot in place of the "a". I indicate the parts in my translations whenever the proof follows this structure.

Abū Kāmil copies not only Euclid's format, but he also rests many of his geometrical proofs on propositions in the *Elements*. He cites Euclid in those propositions whose proofs rely on more than just a quick

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[10] See [Euclid 1956 vol. I, 129; Netz 1999, 10]. I use Netz' translations of the terms. In some proofs one or more of these parts is omitted.

[11] The geometrical proofs are those in category (2), on the products of monomials and binomials, the first two proofs in category (3), on operations with roots, and (6e), the proof to the proposition in problem (61). In these proofs he merges the "construction" into the "proof". The arithmetical proof (6d), found in the margin in problem (7), is not given quite the same format. It may have been added by a copyist.

examination of the different rectangles in the diagram. He even indicates ahead of time that he will be appealing to Euclid just before stating the rules for solving composite equations: "And we will provide all of them, and we show their cause [i.e. we prove them] by geometric figures [which can be] understood by geometers who have seen Euclid's book."[12]

Backing up to the early ninth c., al-Khwārizmī reveals his acquaintance with Greek geometry by including lettered diagrams in his proofs, but he does not follow Euclid's format. Instead, he disregards the parts of the Euclidean proof, most noticeably by omitting the "setting-out" and the "definition of goal". In fact, he makes no references at all to Euclid, instead resting his arguments on an intuitive manipulation of areas.

Abū Kāmil's chapter on the solutions to the six simplified equations is adapted from al-Khwārizmī's book. He lists the equations in the same order as al-Khwārizmī, and he solves all five of al-Khwārizmī's sample composite equations and two of his simple equations as well. He even begins the first rule for the first composite equation with an explicit reference: "So the way to find the root of the *māl* was related by Muḥammad ibn Mūsā al-Khwārizmī in his book, which is that we halve the roots…"[13] In the proofs in this chapter Abū Kāmil pays heed to both Euclid and al-Khwārizmī. He prefers to base his proofs on *Elements* II.5 or II.6, but for the rules to find the "root" in composite equations he gives a second proof in al-Khwārizmī's intuitive manner. In Appendix B I translate the first two proofs for the type 4 equation to illustrate the two approaches.

Abū Kāmil's arithmetical proofs of general results do not exhibit such a mix of influences. Al-Khwārizmī did not give proofs for rules like these, and in fact there is no trace of his method there. Instead, Euclid's influence is pervasive. The proofs follow Euclid's format, and the diagrams are inspired by Euclid's propositions on number theory, comprising Books VII-IX of the *Elements*. To represent arbitrary

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[12] [Abū Kāmil 1986, 7;7; 2004 21;1].
[13] [Abū Kāmil 1986, 7;8; 2004, 21;3].

numbers in these proofs Euclid did not have recourse to our modern algebraic symbolism, so he represented the numbers by lines, which he typically labeled *A*, *B*, etc. Euclid was able to reinterpret the arithmetical operations of addition and subtraction through the concatenation of lines in the diagram. He did this by labeling three or more points on a line. The diagram for *Elements* VII.1, for instance, shows a line with endpoints *C* and *D*, with point *G* between them. Euclid subtracts the number represented by *DG* from the number *DC* to get *GC*. This works because lines, like numbers, are closed under addition and subtraction.

The same is not true of multiplication, division, and roots. The product of two numbers is another number, but the product of two lines jumps a dimension to become a rectangle. So Euclid showed the product of two numbers represented by lines *A* and *B* as a third line *G*, and not by the rectangle formed by *A* and *B*. The diagrams in *Elements* VII-IX are a collection of disjoint lines, with no two-dimensional relation to one another. The diagrams in Abū Kāmil's arithmetical proofs reflect Euclid's practice. Addition and subtraction are represented by concatenation, while the results of other operations are represented by entirely new lines.

Despite the Euclidean form of Abū Kāmil's arithmetical proofs, he does not cite Euclid in them. Abū Kāmil cannot place Arabic arithmetic/algebra on a Euclidean foundation the way he did in his geometrical proofs because his numbers are incompatible with Greek arithmetic. The numbers that form the foundation of Arabic algebra consist of all positive quantities that can arise in calculation, including fractions and irrational roots. By contrast Euclid's arithmetical unit is indivisible, so his number system is restricted to positive integers. Euclid's arithmetical propositions thus cover basic number theory, and not the rules of calculation which are the concern of Abū Kāmil.

Abū Kāmil's adoption of the format of Euclid's proofs and the fact that he rests many of his geometrical proofs on Euclid's propositions indicate that he sought to ground Arabic algebra in the framework of the *Elements*. But supposing that he understood from reading Euclid the need to offer arithmetical proofs in arithmetic and algebra, why did he allow himself to translate specific rules and calculations to a geometrical setting? To answer this I believe we must look beyond the mathematics. The large number of worked-out problems in the *Algebra* tells us that Abū Kāmil's readers were mainly students learning algebra for their occupation, and not theoretical mathematicians. This fact, together with the force of al-Khwārizmī's tradition, would have made geometrical proofs quite attractive. Additionally, the rules for solving simplified equations are complex enough to make an arithmetical proof difficult to follow. They involve a number and its square root in homogeneous roles, a situation that can be nicely handled through metric geometry.

## 6. The exception

The first proof to one general proposition in arithmetic violates Abū Kāmil's convention. Embedded in problem (61) is the rule that if $a - n\sqrt{a} = b + n\sqrt{b}$, then $\sqrt{a} = \sqrt{b} + n$, for $n = 1, 2, 3, \ldots$ Where Abū Kāmil's other numerical propositions are all elementary results in arithmetic, this one is a non-trivial discovery which I have not seen in any other Arabic book. Contrary to his usual approach for propositions stated in general terms, Abū Kāmil gives a geometrical proof for this rule. Even more remarkable is that he follows it up with a second proof, by *al-jabr*. It is the only proof by algebra I have seen in all of medieval mathematics.

> [*protasis* (enunciation)]
> Suppose that for each of two different numbers, if one subtracts from the larger its root, and one adds to the smaller its root, then they are equal. Then the root of the larger number is larger than the root of the smaller number by one, since he said a root and a root. And if he said one subtracts from the larger two of its roots and one adds to the smaller two of its roots, then they are equal. Then the root of the larger number is larger than the root of the smaller by two. And if he

said three roots and three roots, then the root of the larger is larger than the root of the smaller by three…[14]

[*ekthesis* (setting out)]
An example of this is that we make the two quantities squares *ABGD* and *HDWZ* on one line, which is line *GDZF*. And we make the larger quantity square *AGBD* and the smaller quantity square *HDWZ.* And we make the larger square, with two of its roots subtracted, equal to the smaller square with two of its roots added. And we determined that a side of the square *ABGD* is longer than a side of *HDWZ* by line *HB*.

[*diorismos* (definition of goal)]
So I say that line *HB* is two.

[*apodeixis* (proof)]
So if it is not two, then it is larger or smaller than two. So we make it not smaller, and we make line *BL* two, and we draw line *LṬ* parallel to *GD*. So surface *AL* is two roots of surface *AD*. And we append to line *HW* two, which is line *WE*, and we complete surface *WF*, so surface *WF* is two roots of square *HZ.* So the surfaces *LG* and *HF* are equal, and line *DL* is longer than line *HD* and line *HD* is equal to line *DZ.*[15] So line *D* is longer than line *DZ*[16] and line *BL* is equal to line *ZF*. So line *BD* is longer than line *DF* and line *BD* is equal to line *DG*. So line *DG* is longer than line *DF* and *DL* is longer than line *DH*. So surface *LG* is larger than surface *HF*. And this contradicts that we already established that they are equal. So *BH* is not larger than two.

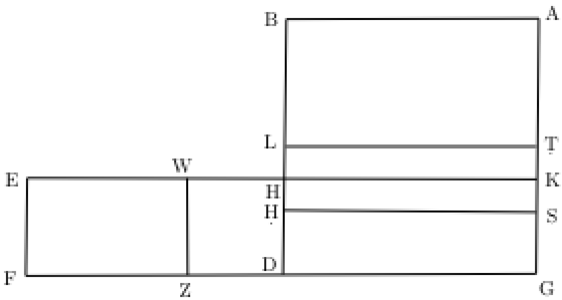
And I say it is not smaller than it. So we declare it so. So we make it smaller than two. And we make *BḤ* two. So surface *AḤ* is

[14] [Abū Kāmil 1986, 122;15; 2004, 198;4].
[15] MS has *GZ*, which is an error.
[16] MS has *HGZ*, which is an error.

two roots of the square *AD*. So surface *SD* is equal to surface *DE* and line *DḤ* is smaller than line *DH* and line *DH* is equal to line *DZ*. So line *DZ* is longer than line *DḤ* and line *ḤB* is equal to line *ZF*, since each one of them is two. So line *DF* is longer than line *DB*, and line *DB* is equal to line *GD*. So line *DF* is longer than line *GD*, and line *GS* is shorter than line *DH*. So surface *DE* is larger than surface *DS*. But they were already equal, so line *BH* is not smaller than two.

And you showed that it is not larger than it, so line *BH* is known to be two. So if we drew line *HK* parallel to line *GD*, then surface *AH* is two roots of square *AD*, and line *BH* is equal to line *ZF*, and *DZ* is equal to *DH*. So *BD* is equal to *DF* and *BD* is equal to *DG*. So *GD* is equal to *DF* and *GK* is equal to *DH*. So surface *KD* is equal to surface *DE*.

[*sumperasma* (conclusion)]
So you have shown that if you subtracted from the larger quantity two of its roots and you added to the smaller two of its roots, and they are equal, then the root of the larger quantity is larger than the root of the smaller by two.[17]




He then comments that this proof works for three roots. Continuing, he writes:

[17] [Abū Kāmil 1986, 123;1; 2004, 198;15].

> Its cause by means of algebra (*al-jabr*) is true and clear. And that is, consider two quantities, where the root of one of them is greater than the root of the other by a dirham. So if you subtracted from the larger its root, and you added to the smaller its root, they became equal, as I mentioned to you. And this is that you make the root of one of the two quantities a thing, and the other a thing and a dirham. So you multiply each one of them by itself, so it yields the smaller is a *māl* and the larger is a *māl* and two things and a dirham. So if you added to the smaller its root, which is a thing, and you subtracted from the larger its root, which is a thing and a dirham, there remained from the larger a *māl* and a thing, and the smaller ended up as a *māl* and a thing, so they are equal.
>
> Likewise if he said we made the smaller a thing and the larger a thing and two dirhams, and we multiplied each one of them by itself, it yielded the smaller is a *māl* and the larger is a *māl* and four things and four dirhams. So if you added to the smaller two of its roots, which are two things, and you subtracted from the larger two of its roots, which are two things and four dirhams, there remained from the larger a *māl* and two things, and the smaller ended up as a *māl* and two things, so they are also equal. And likewise for whatever is larger or smaller than this, return to what we said before that they are equal by adding and subtracting the roots.[18]

It would have been possible for Abū Kāmil to have converted his geometrical proof into an arithmetical proof with disjoint lines, but the complexity of the argument would have made it very difficult to follow. This is a likely reason that he settled for a geometrical proof. We might then see his addition of the algebraic proof as an attempt to patch his failure to make an arithmetical proof. Algebra, after all, is founded on arithmetic.

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[18] [Abū Kāmil 1986, 124;7; 2004, 200;18].

But there may be more to Abū Kāmil's second proof than this simple conjecture. The proposition in problem (61) also happens to be the only one in Abū Kāmil's book which is amenable to an algebraic approach. While other propositions are easily expressed and solved using modern notation, they are not open to proof by medieval algebra.

In order to show why algebraic proofs for the other rules would have been a daunting endeavor, I should say a few words about medieval polynomials. An expression like Abū Kāmil's "a hundred dirhams and two *māl*s less twenty things" ($100 + 2x^2 = 20x$, from problem (2)) was not regarded as a modern linear expression with the operations of addition, subtraction, and multiplication. Instead it was read as a kind of inventory, listing how many of each of the powers are present or lacking. "A hundred dirhams and two *māl*s" is merely a collection of 102 objects of two kinds. It is like saying "a hundred apples and two pears". The "less twenty things" tells us that this amount is missing from those 102 objects, like saying "a dollar less three cents" for the more common English phrase "three cents short of a dollar". Ideally the two sides of an Arabic equation are polynomials. Said another way, they are aggregations of the powers of the unknown linked together not by arithmetical operations, but by the common conjunctions *wa* ("and") and *illā* ("less"). In an Arabic equation the two sides should be numbers, not expressions with quantities left undetermined by the presence of an operation.[19]

When building polynomials in the beginning of the solution to a problem, it is a simple matter to add, subtract, and multiply terms. The multiplication of "a thing" by "ten less a thing" yields the polynomial "ten things less a *māl*", for example. But the set of polynomials is not closed under division, so it is not always easy to set up a polynomial equation for questions that ask for this operation.[20] One of the ways Abū Kāmil solves

[19] See [Oaks 2009] for a detailed account of Arabic polynomials and equations.
[20] I gave examples of the lengths Abū Kāmil went to avoid divisions in equations in [Oaks 2009, §6.4]. One I did not mention there are the rules, with geometrical proofs, he concocted for problems (4) to (7). See Appendix A, category (5) for an example.

this dilemma is to name the results of divisions, and then to use these names in his equations. In problems (2), (23), and (27) he names the result of dividing "a thing" by "ten less a thing" a *dinār*, and its reciprocal a *fals* (these are denominations of coins). Only in problem (59) does he work with terms expressed with a division, starting with "ten dirhams divided by a thing" ($\frac{10}{x}$). Beginning with al-Karajī (early 11th c.) algebraists became more open to expressions of the form "*a* divided by *b*" in equations, but such phrases never became commonplace.

The aversion to operations in expressions is not the only way medieval algebra differs from modern algebra. Arabic algebraists almost always dealt with a single independent unknown, though occasionally one encounters problems solved with multiple named unknown numbers. Abū Kāmil did this in his book *Rarities in Arithmetic*.[21] In addition to "thing", other unknowns are named *dinār*, *fals*, and *khātam*. The last word means "seal ring", or "stamp". I have only seen multiple independent unknowns in problems that yield a linear equation.

At its core Arabic algebra was intended as a means of setting up and solving polynomial equations in one unknown. The key advantage of algebra as a problem solving technique lies with one's ability to combine polynomial expressions to set up an equation, and to simplify the equation to a standard form. Even when algebraists occasionally extended the method to include divisions in equations or to work with more than one unknown, the setting up of the (nearly) polynomial equation and its simplification remained the focus.

In order to tie this in with Abū Kāmil's proofs, consider another proposition, one which is easily written in modern algebraic notation as $\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd}$. Abū Kāmil states it without naming any of the numbers: "And I say also that for any two numbers, if one divides each one of them by a number and multiplies what results from the divisions of the two numbers by two numbers, one of them by the other, then the result of the multiplication is equal to the result of dividing the product of the two divided numbers, one of them by the other, by what results from

[21] [Chalhoub 2001; Sesiano 2000, 149].

multiplying the two dividends, one of them by the other."[22] In his arithmetical proof Abū Kāmil represents the four original numbers with lines labeled *A*, *B*, *G*, and *D*, and the results of multiplications and divisions are represented by new lines *Ḥ*, *L*, *M*, *N*, and *H*. If Abū Kāmil were to give an algebraic proof he would have to name several independent unknowns *and* somehow deal with the divisions. Worse, the advantage of algebra would be lost, since there would be no polynomials to manipulate. While an algebraic proof is possible, the arithmetical proof which Abū Kāmil provides would certainly be clearer.

Algebra would also be an ineffective method for proving the proposition in problem (61), at least the way it is stated. Abū Kāmil would need to name two independent numbers, like "thing" and *dinār*, and to produce two expressions, "a *māl* less a thing" and "a *dinār-māl* (?) and a *dinār*" and somehow perform operations to show that a "thing" is a "*dinār* and a dirham". Instead, he takes a simpler route: he proves the converse. By letting the smaller root be a "thing", he can name the larger root "a thing and a dirham", thus avoiding two names. It is then a simple matter to add to the smaller its root, and to subtract from the larger its root, to show that they are equal. In this proof Abū Kāmil takes advantage of the ease of operating with polynomials which made algebra a successful problem-solving technique.

I wrote about this algebraic proof in [Oaks 2007], where I suggested that medieval algebra could have become "a tool for investigating the relationships of general quantities on par with geometry", but that "[g]eometry was already the established setting for proof in mathematics, so there was no good motive for reworking theories in algebraic form."[23] I see now that the limitations of medieval algebra just described are more likely the reason that we do not find other proofs by algebra. The converse of Abū Kāmil's proposition in problem (61) is open to algebraic proof, but the vast majority of propositions are not.

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22 [Abū Kāmil 1986, 128;5; 2004, 207;3].

23 [Oaks 2007, 569]. By "geometry" I was speaking about any proof accompanied by a diagram, including arithmetical proofs.

Abū Kāmil's proposition in problem (61) is an unusual result proven in unusual ways. He does not explain his approach, so I can only suggest possible motives. I see his geometrical proof as a way to compensate for the difficulty which an arithmetical proof would pose. Abū Kāmil was lucky that algebra was a viable option for this one proposition, albeit via the converse. He was thus able to ground his argument in arithmetic. Neither proof is ideal, but we must commend Abū Kāmil for coming up with such an interesting result and for exploring new ground in his proofs.

## 7. Conclusion

Abū Kāmil deliberately differentiated between propositions stated in general terms and those stated as specific examples by giving the former arithmetical proofs and the latter geometrical proofs. He most likely learned these two kinds of proof from Euclid and al-Khwārizmī respectively. Both the text and the diagrams of Abū Kāmil's arithmetical proofs are structured like Euclid's number theory propositions, and his geometrical proofs are a continuation of al-Khwārizmī's legacy, beginning with the rules for solving simplified equations.

The influence of Euclid extends even into the geometrical proofs, where Abū Kāmil often rests his arguments on specific propositions from the *Elements*. But if he wished to provide a Euclidean foundation for Arabic algebra, he was thwarted by the fact that Arabic numbers cannot be identified with either Euclid's numbers or magnitudes. Numbers in the *Elements* are restricted to positive integers, while Arabic numbers include fractions and irrational roots. Because Euclid's propositions deal with elementary number theory, they are of no use in Abū Kāmil's proofs of rules of calculation. The incompatibility of Arabic numbers with Greek magnitudes lies with the operations of multiplication, division, and roots. Numbers are closed under these operations, while magnitudes change dimension. In a proof one should not reinterpret these arithmetical operations as geometric operations, so the accompanying diagram should not be two dimensional.

In the end Abū Kāmil compromised with both kinds of proof. His geometrical proofs build on the geometry of the *Elements*, but at the expense of the questionable identification of numbers with magnitudes. His arithmetical proofs, though more rigorous, do not refer back to any proposition in the *Elements*.

Other algebraists after the generation of al-Khwārizmī faced the same dilemma as Abū Kāmil: how to reconcile Arabic arithmetic with Greek mathematics. I will mention just two examples to show the different directions algebraists took. Al-Khayyām (late 11th c.) solved the problem by identifying Arabic numbers with the quality of "quantity" shared by continuous magnitudes of different dimensions. So the Arabic number "three and a half" becomes the abstract, dimensionless measure of those lines, planes, and bodies which are three and a half times as large as the unit measure.[24] In this way he was able to securely place algebra on a Euclidean foundation. Ibn al-Bannā' (late 13th c.) was one who took the opposite approach. In his *Book of Fundamentals and Preliminaries in Algebra*[25] he purged Euclid from algebra, and grounded his proofs and explanations solely in terms of Arabic arithmetic.

Abū Kāmil's convention of giving arithmetical proofs to general rules and geometrical proofs to specific examples was maintained for all propositions but one. The general rule embedded in problem (61) is given a geometrical proof, perhaps because an arithmetical proof would have been too difficult. It is followed by a possibly unique proof (of the converse!) by algebra. I see these two proofs as Abū Kāmil's attempts to deal with a difficult and interesting proposition which does not fit with the elementary rules of calculation which receive arithmetical proofs.

Euclid's *Elements* and al-Khwārizmī's *Algebra* were compiled in environments far removed from each other. Abū Kāmil was one of a few mathematicians who attempted to integrate these traditions. What makes his experiment fascinating is not so much the degree to which he was

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[24] [Oaks 2011].

[25] *Kitāb al-uḍūl wa'l-muqaddamāt fī'l-jabr wa'l-muqābala* [Saidan 1986; Djebbar 1981, 25ff].

successful, but the ways he adjusted and compensated for the incongruity of Greek and Arabic mathematics. In the end the tug-and-pull between the legacies of Euclid and al-Khwārizmī resulted in an uneasy stalemate, and Abū Kāmil's book emerges as an emulsion text in which disparate elements of Greek and Arabic mathematics are artificially combined.

## Appendix A. Propositions proven in Abū Kāmil's *Algebra*

(1) Rules for solving simplified equations. (geometrical proofs)

- (a) $x^2 = 5x$, solve for $x$
- (b) $x^2 + 10x = 39$, solve for $x$ (2 proofs)
- (c) $x^2 + 10x = 39$, solve for $x^2$
- (d) $x^2 + 21 = 10x$, solve for $x$ by addition (2 proofs)
- (e) $x^2 + 21 = 10x$, solve for $x$ by subtraction (2 proofs)
- (f) $x^2 + 25 = 10x$ solve for $x$
- (g) $x^2 + 21 = 10x$, solve for $x^2$ by addition
- (h) $x^2 + 21 = 10x$, solve for $x^2$ by subtraction
- (i) $3x + 4 = x^2$, solve for $x$ (3 proofs)
- (j) $3x + 4 = x^2$, solve for $x^2$

(2) Products of algebraic monomials and binomials. (geometrical proofs)

- (a) $2x \cdot 2x = 4x^2$
- (b) $3x \cdot 6 = 18x$
- (c) $(10 + x) \cdot x = 10x + x^2$
- (d) $(10 - x) \cdot x = 10x = x^2$
- (e) $(10 + x) \cdot (10 + x) = 100 + x^2 + 20x$
- (f) $(10 - x) \cdot (10 - x) = 100 + x^2 - 20x$
- (g) $(10 + x) \cdot (10 - x) = 100 - x^2$
- (h) $(10 + x) \cdot (x - 10) = x^2 - 100$

(3) Calculations with square roots. (geometrical proofs, except (d), (e), and (f), which have arithmetical proofs)

(a) $2\sqrt{16} = \sqrt{64}$
(b) $\frac{2}{3}\sqrt{9} = \sqrt{4}$
(c) $\sqrt{9} \cdot \sqrt{4} = \sqrt{36}$
(d) $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$
(e) $\frac{a}{b} \cdot b = a$
(f) $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
(g) $\sqrt{9} + \sqrt{4} = 5$ (by $\sqrt{a} + \sqrt{b} = \sqrt{(a+b) + 2\sqrt{ab}}$ .)
(h) $\sqrt{9} - \sqrt{4} = 1$ (by $\sqrt{a} - \sqrt{b} = \sqrt{(a+b) - 2\sqrt{ab}}$ .)

(4) Algebraic solutions to problems (T1) – (T6). (geometrical proofs)
The enunciations to problems are not stated in algebraic terms, so it would be misleading to give them an algebraic characterization. I give the enunciation to problem (T5) as an example: "Ten: you divided it into two parts. So you multiplied one of them by the other, so it yielded twenty-one." The proof roughly follows the setting up of the equation "ten things except a *māl* equals twenty-one dirhams" ($10x - x^2 = 21$) and its simplification and solution.

(5) Rules to set up polynomial equations in problems (4) – (7). (geometrical proofs)
Again, the enunciations are not given in algebraic terms. The enunciation to problem (4) is: "You divided fifty dirhams among [some] men. So [each] one of them got something. Then you added to them three men. Then you divided among them the fifty dirhams. So each one of them got less than [what] the former [men] got by three dirhams and a half and a quarter of a dirham.[26]

Abū Kāmil will make the number of original men a thing in his solution. But this would yield an equation with divisions: $\frac{50}{x} = \frac{50}{x+3} + 3\frac{3}{4}$. To avoid

[26] [Abū Kāmil 1986, 57;9; 2004, 91;2].

divisions in the equation, he proves that the enunciation can be reformulated by this rule, which he proves by geometry: "Its rule is that you multiply the first men by the difference between what the first men got and what the second men got. So what is gathered from the multiplication, divide it by the difference between the first men and the latter. So what results, multiply it by the latter men. So what is gathered is the divided number [i.e. the fifty dirhams]."[27] This quickly yields the polynomial equation $1\frac{1}{4}x^2 + 3\frac{3}{4}x = 50$, which is free of divisions.[28]

(6) Arithmetic theorems. (arithmetical proofs except (e), which has both a geometrical and an algebraic proof)

(a) $\frac{a}{b} = \frac{a^2}{ab}$ problem (2)

(b) $\frac{a}{b} + \frac{b}{a} = \frac{a^2+b^2}{ab}$ problem (2)

(c) $\frac{a}{b} \cdot \frac{b}{a} = 1$ problem (2)

(d) $a \cdot (\frac{c}{a} - \frac{c}{b}) \cdot b = (b-a) \cdot c$ problem (7)

(e) $a - n\sqrt{a} = b + n\sqrt{b} \Rightarrow \sqrt{a} = \sqrt{b} + n$, for $n = 1$, 2,… problem (61)

(f) $a + b = A \Rightarrow \frac{A}{a} = 1 + \frac{b}{a}$ problem (62)

(g) $a + b = A \Rightarrow \frac{A}{a} \cdot \frac{A}{b} = \frac{A}{a} + \frac{A}{b}$ problem (63)

(h) $\frac{a}{c} \cdot \frac{b}{d} = \frac{a \cdot b}{c \cdot d}$ problem (63)

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27 [Abū Kāmil 1986, 57;11; 2004, 91;6].

28 For an explanation of the aversion of some Arabic algebraists to division in equations, see [Oaks 2009, §6].

## Appendix B. Two proofs for finding the "root" for the type 4 equation.

Abū Kāmil gives two proofs for finding the "root" ($x$) in the equation $x^2 + 10x = 39$. The first proof relies on *Elements* II.6, and the second proof, building on the same diagram, is a "visual" proof similar to those found in al-Khwārizmī.

> And for the *māl*s and the roots which equal the number. So it is like [someone] said to you: a *māl* and ten roots equal thirty-nine dirhams…[29]
>
> So the way to find the root of the *māl* was related by MuḤammad ibn Mūsā al-Khwārizmī in his book, which is that we halve the roots, which are in this problem five. So we multiply it by itself, so it yields twenty-five. So we add it to the thirty-nine, so it yields sixty-four. So we take its root, so it yields eight. So subtract from it half the roots, which is five. So there remain three, which is the root of the *māl*, and the *māl* is nine…[30]
>
> [*First proof for finding the root*]
>
> And for the cause of "a *māl* and ten roots equal thirty-nine". So the way to find the root is that we make the *māl* the square surface *ABGD*, and we attach to it the roots which are with it, which are ten roots, which is surface *ABWH*. So it is clear that line *BH* is ten in number, since a side of surface *ABGD*, which is line *AB*, multiplied by one yields the root of the surface *ABGD*. So this multiplied by ten yields ten roots of the surface *ABGD*. So line *BH* is known to be ten.
>
> And surface *WHGD* is thirty-nine, since it is a *māl* and ten roots, which comes from the product of line *HG* by line *GD*. And line *GD* is equal to line *GB*. So the product of line *HG* by line *GB* is thirty-nine, and line *HB* is ten.

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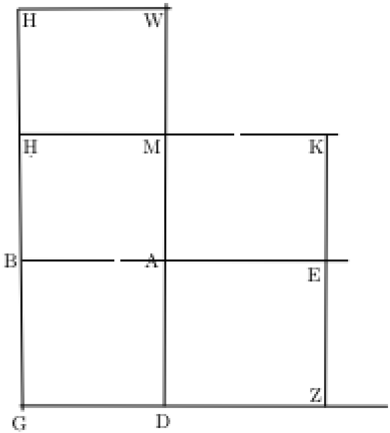
[29] [Abū Kāmil 1986, 7;3; 2004, 20;16].

[30] [Abū Kāmil 1986, 7;8; 2004, 21;3].

So we divide line *HB* into two halves at point *Ḥ*. So line *HB* has been divided into two halves at point *Ḥ*, and is extended in length by line *BG*. So the product of line *HG* by line *GB* and Ḥ*B* by itself is equal to the product of line *ḤG* by itself, as Euclid said in Book II of his work.[31] And the product of line *HG* by line *GB* is thirty-nine, and the product of line *ḤB* by itself is twenty-five. So the product of line *ḤG* by itself is sixty-four. So line *ḤG* is eight, and line *ḤB* is five. So line *BG*, the remainder, is three, which is the root of the *māl*, and the *māl* is nine.

[*Second proof for finding the root*]
And if you wished to visualize what was said to you: we constructed on line *ḤG* a square surface which is surface *ḤKZG*, and we draw line *AB* straight to point *E*. So line *ḤG* is equal to line *GZ*, and line *BG* is equal to line *GD*. So the remaining line *BḤ* is equal to line *DZ*. So surface *ḤA* is equal to surface *AZ*, and surface *ḤA* is equal to surface *MH*. So surface *MH* is equal to surface *AZ*. So the three surfaces *MB*, *BD*, *DE* are thirty-nine, and surface *AK* is twenty-five, since it comes from the product of line *ḤB* by itself, and line *ḤB* is five, so surface *AK* is twenty-five. So the whole surface *KG* is sixty-four. So line *GḤ* is its root, which is eight, and line *ḤB* is five. So line *BG*, the remainder, is three, and that is what we wanted to show.[32]

[31] *Elements* II.6.
[32] [Abū Kāmil 1986, 10;5; 2004, 24;10].

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